

Properties of hyperkähler manifolds and their twistor spaces

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ABSTRACT

We describe the relation between supersymmetric σ -models on hyperkähler manifolds, projective superspace, and twistor space. We review the essential aspects and present a coherent picture with a number of new results.

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1 Introduction and a succinct mathematical summary

This paper collects the insights that we have gained over twenty years of studying supersymmetric σ -models and hyperkähler geometry. Many of our results have appeared elsewhere, both in our work and in the work of others. Here we want to present a coherent view of how supersymmetry naturally reveals the geometric structure; in particular, we are led to the twistor spaces of hyperkähler manifolds.

Supersymmetric σ -models are described by an action functional for maps from a space-time into a target manifold; we focus on the case when the target space of the σ -models is hyperkähler [1]. Supersymmetry is most naturally studied by extending the spacetime to a superspace with fermionic as well as bosonic dimensions. $N = 2$ supersymmetric σ -models in four spacetime dimensions (as well as their dimensional reductions in three and two dimensions)¹ are best described in projective superspace² [2], [3], [4], [5], [6], [7], [8], [9], [10], [11]. Projective superspace naturally leads to twistor space [12, 13, 14, 8, 16].

We begin with a brief mathematical summary of some of our main results that also serves as an introduction to hyperkähler geometry. A hyperkähler space \mathcal{M} supports three globally defined integrable complex structures I, J, K obeying the quaternion algebra: $IJ = -JI = K$, plus cyclic permutations. Any linear combination of these, $aI + bJ + cK$ is again a Kähler structure on \mathcal{M} if $a^2 + b^2 + c^2 = 1$, *i.e.*, if $\{a, b, c\}$ lies on a two-sphere $S^2 \simeq \mathbb{P}^1$. The Twistor space \mathcal{Z} of a hyperkähler space \mathcal{M} is the product of \mathcal{M} with this two-sphere $\mathcal{Z} = \mathcal{M} \times \mathbb{P}^1$. The two-sphere thus parameterizes the complex structures and we choose projective (inhomogeneous) coordinates ζ to describe it (in a patch including the north pole). A choice of ζ corresponds to a choice of a preferred complex structure, *e.g.*, J . The corresponding Kähler form $\omega^{(1,1)}$ is a $(1,1)$ two-form with respect to J . For this choice, the two remaining independent complex structures I and K can be used to construct the holomorphic and antiholomorphic symplectic two-forms $\omega^{(2,0)}$ and $\omega^{(0,2)}$. These three two-forms are conveniently combined into [14]

$$\Omega(\zeta) \equiv \omega^{(2,0)} + \zeta \omega^{(1,1)} - \zeta^2 \omega^{(0,2)} , \quad (1.1)$$

which is a section of a two-form valued $\mathcal{O}(2)$ bundle on \mathbb{P}^1 . For the four-dimensional case, the statement that the hyperkähler space obeys the Monge-Ampère equation,

$$2 \omega^{(2,0)} \omega^{(0,2)} = (\omega^{(1,1)})^2 , \quad (1.2)$$

¹The formalism can also be developed in six dimensions [4] as well as five dimensions [15]; however, the four dimensional formalism is the most familiar.

²The terminology “projective superspace” is historic; we are not actually considering a projective supermanifold.

simply becomes the identity³

$$\Omega^2 = 0 . \quad (1.3)$$

For higher-dimensional manifolds the corresponding identity

$$\Omega^{n+1} = 0 \quad (1.4)$$

results in a system of equations constraining the geometry to be hyperkähler.

For $\zeta = 0$, we can choose local Darboux coordinates (holomorphic with respect to the complex structure at the north pole) for $\omega^{2,0}$; as we smoothly rotate the \mathbb{P}^1 of complex structures, we find Darboux coordinates $\Upsilon^p(\zeta)$ and $\tilde{\Upsilon}_p(\zeta)$ that are regular at $\zeta = 0$:

$$\Omega(\zeta) = i d\Upsilon^p(\zeta) d\tilde{\Upsilon}_p(\zeta) \quad (1.5)$$

where $p = 1, \dots, n$ and the (real) dimension of \mathcal{M} is $4n$, and the exterior derivative acts only along \mathcal{M} and *not* along the \mathbb{P}^1 . We introduce the real-structure \mathfrak{R} on \mathbb{P}^1 defined by complex conjugation composed with the antipodal map. From (1.1) we see that the two-form Ω obeys the reality condition

$$\Omega(\zeta) = -\zeta^2 \mathfrak{R}(\Omega(\zeta)) ; \quad (1.6)$$

since

$$\mathfrak{R}(\Upsilon^p(\zeta)) = \tilde{\Upsilon}^p(-\frac{1}{\zeta}) \quad (1.7)$$

we have

$$i d\Upsilon^p(\zeta) d\tilde{\Upsilon}_p(\zeta) = i \zeta^2 d\tilde{\Upsilon}^p(-\frac{1}{\zeta}) d\tilde{\tilde{\Upsilon}}_p(-\frac{1}{\zeta}) \quad (1.8)$$

The reality relations (1.6,1.8) show that Υ and $\tilde{\Upsilon}$ are related to $\bar{\Upsilon}$ and $\bar{\tilde{\Upsilon}}$ by a symplectomorphism up to the ζ^2 -factor. We introduce a generating function $f(\Upsilon, \tilde{\Upsilon}; \zeta)$ for this twisted symplectomorphism:

$$\tilde{\Upsilon}_p = \zeta \frac{\partial f}{\partial \Upsilon^p} , \quad \bar{\tilde{\Upsilon}}_p = -\frac{1}{\zeta} \frac{\partial f}{\partial \Upsilon^p} ; \quad (1.9)$$

then

$$i d\Upsilon^p d\tilde{\Upsilon}_p = i \zeta \frac{\partial^2 f}{\partial \Upsilon^p \partial \Upsilon^q} d\Upsilon^p d\bar{\Upsilon}^q \equiv i \zeta \partial \bar{\partial} f , \quad (1.10)$$

where ∂ and $\bar{\partial}$ are respectively holomorphic and anti-holomorphic exterior derivatives with respect to the complex structure J at the north pole of the \mathbb{P}^1 , and again act only on \mathcal{M} and not along the \mathbb{P}^1 . The conditions above imply that $\zeta \frac{\partial f}{\partial \Upsilon^p}$ is regular at the north pole, and hence, for a contour encircling $\zeta = 0$,

$$\oint \frac{d\zeta}{2\pi i \zeta} \zeta^i \frac{\partial f}{\partial \Upsilon^p} = 0 , \quad i \geq 2 , \quad (1.11)$$

³For the four dimensional case, these ideas were found previously in a different context [17].

as well as the complex conjugate relation. As we shall see in subsequent sections, this beautiful mathematics follows from the σ -model. In particular, (1.11) are the equations of motion, and f is the projective superspace Lagrangian. Thus the function f has the rôle both of the superspace σ -model Lagrangian and as a generating function for north-south symplectomorphisms.

One of our new observations generalizes a result proven in [14] for the special case when the rotation of the complex structures is generated by an isometry of the manifold. In general, rotations of the sphere of complex structures correspond to nonholomorphic diffeomorphisms on the hyperkähler manifold. In twistor space we can compose such a rotation with the corresponding diffeomorphism to construct a symplectomorphism preserving Ω (up to the ζ factor). Going to Darboux-coordinates for $\omega^{(2,0)}$ we can analyze the effect of these rotations on the Kähler potential K . It does not transform simply under rotations of the complex structures but the net result is always a new \tilde{K} . We find that for any hyperkähler manifold, the *moment map* for transformations with respect to rotations about an axis is the *Kähler potential* with respect to any complex structure in the equatorial plane normal to the axis.

2 Review of projective superspace and SUSY σ -models

The projective superspace⁴ approach to $N = 2$ supersymmetry has been discussed many times [2, 3, 14, 6, 8]; a concise but extensive review can be found in the appendices of [20]. Here we review the aspects relevant to this paper.

We want to emphasize that the requirements of supersymmetry in spacetime naturally lead to the constructions that we describe, and lead us to uncover the geometric structures of the target space.

2.1 Spinor derivatives

Superspace is a space with both bosonic and fermionic coordinates; its essential properties are captured in the algebra of the fermionic derivatives. The algebra of $N = 2$ superspace derivatives in four (spacetime) dimensions is

$$\{D_{a\alpha}, D_{b\beta}\} = \{\bar{D}_{\dot{\alpha}}^a, \bar{D}_{\dot{\beta}}^b\} = 0, \quad \{D_{a\alpha}, \bar{D}_{\dot{\beta}}^b\} = i\delta_a^b \partial_{\alpha\dot{\beta}}, \quad (2.1)$$

where $a, b = 1 \dots 2$ are isospin indices and α, β and $\dot{\alpha}, \dot{\beta}$ are left and right handed spinor indices respectively. Mathematically, the D 's are Grassmann odd derivations that are sections of the self-dual spin-bundle tensored with an associated $SU(2)$ bundle, $S_+ \otimes \mathbb{C}^2$,

⁴A related [18] formalism is harmonic superspace, as described in [19] and references therein.

and the \bar{D} 's are sections of $S_- \otimes \mathbb{C}^2$. The superspace derivatives $D_{1\alpha}, \bar{D}_{\dot{\alpha}}^1$ generate an $N = 1$ subalgebra; we will often decompose representations of the full $N = 2$ algebra in terms of $N = 1$ representations.

We may parameterize a \mathbb{P}^1 of maximal graded abelian subalgebras as⁵

$$\nabla_{\alpha}(\zeta) = D_{2\alpha} + \zeta D_{1\alpha} , \quad \bar{\nabla}_{\dot{\alpha}}(\zeta) = \bar{D}_{\dot{\alpha}}^1 - \zeta \bar{D}_{\dot{\alpha}}^2 , \quad (2.2)$$

where ζ is the inhomogeneous coordinate on \mathbb{P}^1 in a patch around the north pole and $\bar{\nabla}_{\dot{\alpha}}(\zeta)$ is the conjugate of $\nabla_{\alpha}(\zeta)$ with respect to the real structure \Re (complex conjugation composed with the antipodal map on \mathbb{P}^1):

$$\bar{\nabla}(\zeta) \equiv -\zeta \Re(\nabla(\zeta)) = -\zeta \nabla^* \left(-\frac{1}{\zeta} \right) . \quad (2.3)$$

2.2 Superfields

Superfields are the generalizations of functions and sections of bundles to superspace. Superfields in projective superspace are by definition annihilated by the projective derivatives (2.2); they differ by their analytic properties on the \mathbb{P}^1 parameterized by ζ . The most general superfield that describes a scalar multiplet is the arctic multiplet Υ , which is analytic around the north pole, and its conjugate antarctic multiplet $\tilde{\Upsilon}$, which is analytic around the south pole [6]. The conjugate is again defined with respect to the real structure \Re . In some cases, we impose a reality condition on Υ . Other useful superfields are tropical; they may have singularities at both poles, but are regular in a region where the two coordinate patches overlap. These are also usually taken to be real.

Because the derivatives $\nabla(\zeta), \bar{\nabla}(\zeta)$ all anticommute, we may impose the conditions

$$\nabla_{\alpha}(\zeta)\Upsilon(\zeta) = \bar{\nabla}_{\dot{\alpha}}(\zeta)\Upsilon(\zeta) = 0 ; \quad (2.4)$$

these imply

$$D_{1\alpha}\Upsilon_{i-1} + D_{2\alpha}\Upsilon_i = \bar{D}_{\dot{\alpha}}^2\Upsilon_{i-1} - \bar{D}_{\dot{\alpha}}^1\Upsilon_i = 0 , \quad (2.5)$$

where

$$\Upsilon = \sum_{i=0} \Upsilon_i \zeta^i . \quad (2.6)$$

The relations (2.5) imply the constraints

$$\bar{D}_{\dot{\alpha}}^1\Upsilon_0 = \bar{D}^{1\dot{\alpha}}\bar{D}_{\dot{\alpha}}^1\Upsilon_1 = 0 . \quad (2.7)$$

If we decompose Υ into its $N = 1$ content, we see that only the coefficients Υ_0, Υ_1 (and their complex conjugates) are constrained as $N = 1$ superfields—the constraints (2.5) do not imply any constraints in $N = 1$ superspace for the remaining coefficients.

⁵In many papers, *e.g.*, [3, 5, 8, 10, 11], the role of D_1 and D_2 are interchanged. However, this leads to inconvenient identifications of the holomorphic coordinates, and we choose conventions compatible with [16].

2.3 SUSY σ -model Lagrangians

Field theories describing maps from a spacetime into a target manifold M are called σ -models, and are generally described by a Lagrangian. The fields map points of spacetime to points of the target M .

The projective superspace Lagrange density F of a σ -model with a real $4D$ -dimensional target M is a contour integral on \mathbb{P}^1 of an unconstrained function $f(\Upsilon^a, \bar{\Upsilon}^a; \zeta)$ of the multiplets Υ^a , $a = 1 \dots D$ as well as the coordinate ζ :

$$F(\Upsilon_i^a, \bar{\Upsilon}_i^a) = \oint_C \frac{d\zeta}{2\pi i \zeta} f(\Upsilon^a, \bar{\Upsilon}^a; \zeta) ; \quad (2.8)$$

the function f is real with respect to the real structure modulo terms that do not contribute to the contour integral, and F is real. For general polar multiplets, since all we know about $\Upsilon, \bar{\Upsilon}$ is that they are analytic near the north and south pole respectively, this is a purely formal expression and the contour C is not yet defined; we will see how to make this into a sensible contour integral below. For other multiplets, the contour depends on f and in known examples turns out to be essentially unique.

The Lagrangian is, *e.g.*,

$$L = D_1^\alpha D_{1\alpha} \bar{D}^{1\dot{\alpha}} \bar{D}_{\dot{\alpha}}^1 F ; \quad (2.9)$$

because of the constraints (2.4), the action $\int d^4x L$ is invariant under the full $N = 2$ supersymmetry.

3 Superspace equations of motion

The equations that describe the extrema of the action can be described in superspace. Since the $N = 2$ Lagrangian is written with an $N = 1$ measure (2.9), the equations of motion that follow from varying with respect to Υ can best be understood by thinking of the $N = 1$ superspace content of the ζ -expansion of Υ . The constraints (2.5, 2.7) for a general polar multiplet imply that as $N = 1$ superfields, all the Υ_i , $i \geq 2$ are unconstrained. The equations that follow from varying them are (we suppress the index a that labels the various Υ superfields):

$$\frac{\partial F}{\partial \Upsilon_i} = \oint_C \frac{d\zeta}{2\pi i \zeta} \zeta^i \left(\frac{\partial}{\partial \Upsilon} f(\Upsilon, \bar{\Upsilon}; \zeta) \right) = 0 \quad , \quad i \geq 2 . \quad (3.1)$$

Here the contour should really be interpreted as enclosing $\zeta = 0$; the auxiliary $N = 1$ superfields $\Upsilon_{i>1}$ are eliminated in such a way as to make sense of this contour. The equations that follow from varying with respect to the constrained $N = 1$ superfields Υ_1 and Υ_0 can be found by applying \bar{D}_α^2 and $\bar{D}^{2\dot{\alpha}} \bar{D}_{\dot{\alpha}}^2$ to the Υ_2 equation and then using the

$N = 2$ constraint (2.4) $\bar{\nabla}_{\dot{\alpha}} \frac{\partial f}{\partial \Upsilon} = 0$ to re-express the equations in terms of $\bar{D}_{\dot{\alpha}}^1$ and $\bar{D}^{1\dot{\alpha}} \bar{D}_{\dot{\alpha}}^1$, respectively.

It is important to distinguish $N = 1$ and $N = 2$ on-shell constraints. When the conditions (3.1) are interpreted in $N = 1$ superspace, they serve only to eliminate unconstrained (auxiliary) $N = 1$ superfields, and so they do not put the $N = 1$ theory on-shell. When we impose $N = 2$ supersymmetry as described in the previous paragraph, field equations for the physical $N = 1$ superfields follow from (3.1), and the theory is fully on-shell.

The equations (3.1) simply imply that $\frac{\partial}{\partial \Upsilon} f(\Upsilon, \tilde{\Upsilon}; \zeta)$ and hence $\partial f \equiv \frac{\partial}{\partial \Upsilon} f(\Upsilon, \tilde{\Upsilon}; \zeta) d\Upsilon$ have at most simple poles; here ∂ is a holomorphic derivative *without* a term $d\zeta \partial_{\zeta}$ along \mathbb{P}^1 and $d\Upsilon \equiv \sum \zeta^i d\Upsilon_i$. Thus when one imposes the equations (3.1),

$$\tilde{\Upsilon} \equiv \zeta \frac{\partial}{\partial \Upsilon} f(\Upsilon, \tilde{\Upsilon}; \zeta) \quad (3.2)$$

is again an arctic multiplet.

The conjugate equation

$$\frac{\partial F}{\partial \Upsilon_i} = \oint_C \frac{d\zeta}{2\pi i \zeta} (-\zeta)^{-i} \left(\frac{\partial}{\partial \Upsilon} f(\Upsilon, \tilde{\Upsilon}; \zeta) \right) = 0 \quad , \quad i \geq 2 \quad (3.3)$$

similarly implies that $\bar{\partial} f(\Upsilon, \tilde{\Upsilon}; \zeta)$ has at most simple zeros. Formally, the equations (3.1,3.3) can be used to eliminate the components $\Upsilon_i, \tilde{\Upsilon}_i, i \geq 2$ in terms of $\Upsilon_0, \Upsilon_1, \tilde{\Upsilon}_0, \tilde{\Upsilon}_1$. Given such a solution, Υ and $\tilde{\Upsilon}$ become maps on \mathbb{P}^1 ; substituting back into (2.8), for a contour that encloses the relevant singularities, the formal expression now becomes well defined. In $N = 1$ superspace, the equations (3.1,3.3) serve to eliminate the $N = 2$ superfields that are unconstrained as $N = 1$ superfields; thus the Lagrangian (2.8) results in a well defined $N = 1$ superspace action for the $N = 1$ superfields $\{\Upsilon_0, \Upsilon_1, \tilde{\Upsilon}_0, \tilde{\Upsilon}_1\}$, or equivalently, for the $N = 1$ (anti)chiral superfields $\{\Upsilon_0, \tilde{\Upsilon}_0, \bar{\Upsilon}_0, \bar{\tilde{\Upsilon}}_0\}$.

4 The $N = 1$ superspace Lagrangian

In $N = 1$ superspace, the σ -model superspace Lagrangian is the Kähler potential expressed as a function of chiral superfields that geometrically are identified as holomorphic coordinates. Here we find the $N = 1$ superspace Lagrangian that arises after solving the equations (3.1,3.3); the Kähler potential can be written in terms of the $N = 1$ (anti)chiral superfields $\{z \equiv \Upsilon_0, u \equiv \tilde{\Upsilon}_0, \bar{z} \equiv \bar{\Upsilon}_0, \bar{u} \equiv \bar{\tilde{\Upsilon}}_0\}$:

$$K(z, \bar{z}, u, \bar{u}) = \oint_C \frac{d\zeta}{2\pi i \zeta} f - u \oint_{O_N} \frac{d\zeta}{2\pi i \zeta} \frac{1}{\zeta} \Upsilon - \bar{u} \oint_{O_S} \frac{d\zeta}{2\pi i \zeta} (-\zeta) \tilde{\Upsilon} \quad (4.1)$$

where $O_{N,S}$ are the contours around the north and south poles; we can write

$$u = \oint \frac{d\zeta}{2\pi i \zeta} \tilde{\Upsilon} \quad , \quad \bar{u} = \oint \frac{d\zeta}{2\pi i \zeta} \bar{\tilde{\Upsilon}} \quad ,$$

$$z = \oint \frac{d\zeta}{2\pi i \zeta} \Upsilon \quad , \quad \bar{z} = \oint \frac{d\zeta}{2\pi i \zeta} \bar{\Upsilon} \quad . \quad (4.2)$$

5 The 2-form Ω and the meaning of the Lagrangian

In this section we construct a 2-form that leads us to a geometric interpretation of the $N = 2$ superspace Lagrangian. As we shall see in subsequent sections, this 2-form captures the essential aspects of hyperkähler geometry.

An essential observation is that (3.1,3.3) imply that

$$\Omega \equiv i\zeta \partial \bar{\partial} f = i\zeta \frac{\partial^2}{\partial \Upsilon^a \partial \bar{\Upsilon}^b} f(\Upsilon, \bar{\Upsilon}; \zeta) d\Upsilon^a d\bar{\Upsilon}^b \quad (5.1)$$

is a section of an $\mathcal{O}(2)$ bundle. The two-form Ω plays a central role in our understanding of the mathematical structure of the model. It can also be written as

$$\Omega = id\Upsilon d\tilde{\Upsilon} = i\zeta^2 d\bar{\Upsilon} d\tilde{\Upsilon} \quad (5.2)$$

where $\tilde{\Upsilon} = -\frac{1}{\zeta} \frac{\partial}{\partial \bar{\Upsilon}} f$. Note that because $\Upsilon, \tilde{\Upsilon}$ are arctic and $\bar{\Upsilon}, \bar{\tilde{\Upsilon}}$ are antarctic, equation (5.2) *implies* that Ω is a section of an $\mathcal{O}(2)$ bundle.

Equation (5.2) has the form of a twisted symplectomorphism, and therefore there should exist a generating function for this transformation. Indeed, (3.2) and its conjugate allow us to identify the $N = 2$ superspace Lagrangian $f(\Upsilon, \bar{\Upsilon}; \zeta)$ as this generating function.⁶

6 Generalized $\Upsilon \leftrightarrow \tilde{\Upsilon}$ duality transformations

Dualities of various sorts have been considered extensively in superspace. A rather trivial kind results in a diffeomorphism on the target manifold. In projective superspace, one may generate such a diffeomorphism by relaxing the regularity constraint on Υ and re-imposing it with an arctic Lagrange multiplier $\tilde{\Upsilon}$:

$$f(\Upsilon, \bar{\Upsilon}; \zeta) \rightarrow f(Y, \bar{Y}; \zeta) - \frac{\tilde{\Upsilon} Y}{\zeta} + \bar{\tilde{\Upsilon}} \bar{Y} \zeta ; \quad (6.1)$$

integrating out $\tilde{\Upsilon}, \bar{\tilde{\Upsilon}}$ imposes the constraints that Y, \bar{Y} are arctic and antarctic respectively; integrating out Y, \bar{Y} gives a dual Lagrangian $f(\tilde{\Upsilon}, \bar{\tilde{\Upsilon}}; \zeta)$ which is the Legendre transform of f . This corresponds to simply interchanging the roles of Υ and $\tilde{\Upsilon}$ above.

⁶Superspace Lagrangians with the interpretation of a generating function of a symplectomorphism have also been discovered in the context of σ -models with bihermitian target spaces [21].

The interpretation of the $N = 2$ superspace Lagrange density f as the generating function of a twisted symplectomorphism from holomorphic coordinates adapted to the complex structure at the north pole to those at the south pole allows us to generalize this duality.

We can construct holomorphic symplectomorphisms of $\Upsilon, \tilde{\Upsilon} \rightarrow \chi, \tilde{\chi}$ and compose them with f to find the transformed $N = 2$ superspace Lagrange densities. Explicitly, we consider a generating function $g(\Upsilon, \chi; \zeta)$ such that

$$\tilde{\Upsilon} = \frac{\partial g}{\partial \Upsilon}, \quad \tilde{\chi} = -\frac{\partial g}{\partial \chi}, \quad (6.2)$$

where the explicit ζ dependence of g is such that $\Upsilon, \tilde{\Upsilon}, \chi, \tilde{\chi}$ are all arctic. By polar conjugation we have $\bar{g}(\bar{\Upsilon}, \bar{\chi}; \bar{\zeta})$ such that

$$\bar{\tilde{\Upsilon}} = \frac{\partial \bar{g}}{\partial \bar{\Upsilon}}, \quad \bar{\tilde{\chi}} = -\frac{\partial \bar{g}}{\partial \bar{\chi}}. \quad (6.3)$$

Then the transformed Lagrange density $h(\chi, \bar{\chi}; \zeta)$ is given by

$$h = f(\Upsilon(\chi, \bar{\chi}; \zeta), \bar{\Upsilon}(\chi, \bar{\chi}; \bar{\zeta}); \zeta) + \frac{1}{\zeta} g(\Upsilon(\chi, \bar{\chi}; \zeta), \chi; \zeta) - \zeta \bar{g}(\bar{\Upsilon}(\chi, \bar{\chi}; \bar{\zeta}), \bar{\chi}; \bar{\zeta}) \quad (6.4)$$

where $\Upsilon(\chi, \bar{\chi}; \zeta), \bar{\Upsilon}(\chi, \bar{\chi}; \bar{\zeta})$ are determined by

$$\frac{\partial g(\Upsilon, \chi; \zeta)}{\partial \Upsilon} = -\zeta \frac{\partial f(\Upsilon, \bar{\Upsilon}; \zeta)}{\partial \Upsilon}, \quad \frac{\partial \bar{g}(\bar{\Upsilon}, \bar{\chi}; \bar{\zeta})}{\partial \bar{\Upsilon}} = \frac{1}{\zeta} \frac{\partial f(\Upsilon, \bar{\Upsilon}; \zeta)}{\partial \bar{\Upsilon}}. \quad (6.5)$$

To check this, we need to see that

$$\tilde{\chi} = -\zeta \frac{\partial h}{\partial \chi}; \quad (6.6)$$

using (6.4), we have:

$$-\zeta \frac{\partial h}{\partial \chi} = -\zeta \left(\frac{\partial f}{\partial \Upsilon} \frac{\partial \Upsilon}{\partial \chi} + \frac{\partial f}{\partial \bar{\Upsilon}} \frac{\partial \bar{\Upsilon}}{\partial \chi} \right) - \frac{\partial g}{\partial \Upsilon} \frac{\partial \Upsilon}{\partial \chi} - \frac{\partial g}{\partial \chi} + \zeta^2 \frac{\partial \bar{g}}{\partial \bar{\Upsilon}} \frac{\partial \bar{\Upsilon}}{\partial \chi}; \quad (6.7)$$

from (6.5), this gives $-\zeta \frac{\partial h}{\partial \chi} = -\frac{\partial g}{\partial \chi}$, and hence, from (6.2), we find (6.6).

7 $\mathcal{O}(2n)$ -multiplets and Killing spinors

In this section, we consider projective superfields that are sections of certain bundles on the \mathbb{P}^1 . In particular, $\Upsilon \equiv \eta_{(2n)}$ may be a section of a $\mathcal{O}(2n)$ bundle⁷ over \mathbb{P}^1 [6, 22]:

$$\Upsilon(\zeta) \equiv \eta_{(2n)}(\zeta) = (-)^n \zeta^{2n} \bar{\Upsilon}(-\frac{1}{\zeta}). \quad (7.1)$$

⁷The $\mathcal{O}(2)$ case is special because it arises for hyperkähler manifolds admitting a triholomorphic torus action, and has been discussed extensively [3, 14].

Thus $\eta_{(2n)}(\zeta)$ is a polynomial of order $2n$ in ζ . We show that σ -models described in terms of these $\mathcal{O}(2n)$ -multiplets admit certain local Killing spinors. These multiplets as well as other special multiplets were considered in [6].

7.1 Supersymmetric σ -models and $\mathcal{O}(2n)$ -multiplets

We begin with a review of $\mathcal{O}(2n)$ -multiplets and the generalized Legendre transform construction [6].

The formal expression for the σ -model Lagrangian (2.8) can be made well-defined without imposing the conditions (3.1,3.3) if we impose certain constraints on Υ . Here we focus on the constraint that Υ is a section of an $\mathcal{O}(2n)$ -bundle. We may then impose the reality condition (7.1):

$$\Upsilon(\zeta) \equiv \eta_{(2n)}(\zeta) = (-)^n \zeta^{2n} \bar{\Upsilon}(\frac{1}{\zeta}) ; \quad (7.2)$$

$\eta_{(2n)}(\zeta) = \sum_{i=0}^{2n} \zeta^i \eta_i$ is a polynomial of order $2n$ in ζ obeying the constraints:

$$\bar{\eta}_i = (-1)^{n-i} \eta_{2n-i} . \quad (7.3)$$

Now we can find a suitable contour (see, *e.g.*, the discussion in [23]) and compute the Lagrange density

$$F(\eta_i) = \oint_C \frac{d\zeta}{2\pi i \zeta} f(\eta; \zeta) ; \quad (7.4)$$

As for the polar case, the Kähler potential is found by eliminating the $N = 1$ auxiliary superfields η_i , $2 \leq i \leq 2(n-1)$ and performing a complex Legendre transform with respect to η_1 and $\eta_{2n-1} = (-1)^n \bar{\eta}_1$:

$$K(z, \bar{z}, u, \bar{u}) = F\left(\eta_i(z, \bar{z}, u, \bar{u})\right) - u \eta_1(z, \bar{z}, u, \bar{u}) - \bar{u} \bar{\eta}_1(z, \bar{z}, u, \bar{u}) , \quad (7.5)$$

where $\eta_i(z, \bar{z}, u, \bar{u})$ are found by solving (preserving the reality conditions (7.3)):

$$z = \eta_0 , \quad u = \frac{\partial F(\eta_i)}{\partial \eta_1} , \quad \frac{\partial F(\eta_i)}{\partial \eta_j} = 0 , \quad 2 \leq j \leq 2(n-1) . \quad (7.6)$$

7.2 Four-dimensional hyperkähler manifolds

We begin by considering 4(real)-dimensional manifolds; the generalization to higher dimensions is given later. We prove that a σ -model model description in terms of a $\mathcal{O}(2n)$ -multiplet is possible if and only if the manifold admits a $2n$ -index Killing spinor⁸.

The metric of a hyperkähler manifold satisfies the Monge-Ampère equation; we can always find holomorphic coordinates such that this has the form

$$K_{u\bar{u}} K_{z\bar{z}} - K_{u\bar{z}} K_{z\bar{u}} = 1 . \quad (7.7)$$

⁸This was shown using different techniques in [24].

This implies that we can write the line element as

$$ds^2 = |kdz|^2 + |k^{-1}du + kK_{z\bar{u}}dz|^2 \quad (7.8)$$

where

$$k \equiv K_{u\bar{u}}^{-\frac{1}{2}} . \quad (7.9)$$

We choose frames $\hat{e}^{A\dot{B}}$ (here A, \dot{B} are *target* space spinor indicies)

$$\hat{e}^{+\dot{+}} = kd\bar{z} \quad , \quad \hat{e}^{+\dot{-}} = k\bar{\partial}K_u = \frac{d\bar{u}}{k} + kK_{u\bar{z}}d\bar{z} , \quad (7.10)$$

$$\hat{e}^{-\dot{+}} = kdz \quad , \quad \hat{e}^{-\dot{-}} = -k\partial K_{\bar{u}} = -\left(\frac{du}{k} + kK_{z\bar{u}}dz\right) , \quad (7.11)$$

(so that $ds^2 = \hat{e}^{+\dot{+}}\hat{e}^{-\dot{-}} - \hat{e}^{+\dot{-}}\hat{e}^{-\dot{+}}$). We compute the connection; it is self-dual, with $\omega^A_B = 0$; the nonvanishing terms are

$$\omega^{\dot{+}}_{\dot{+}} = -\omega^{\dot{-}}_{\dot{-}} = (\bar{\partial} - \partial)\ln(k) , \quad \omega^{\dot{+}}_{\dot{-}} = K_{u\bar{u}}\bar{\partial}\left(\frac{K_{z\bar{u}}}{K_{u\bar{u}}}\right) , \quad \omega^{\dot{-}}_{\dot{+}} = -K_{u\bar{u}}\partial\left(\frac{K_{u\bar{z}}}{K_{u\bar{u}}}\right) \quad (7.12)$$

The dual vector fields are

$$e^{\dot{-}}_{\dot{+}} = -k^{-1}\partial_{\bar{z}} + kK_{u\bar{z}}\partial_{\bar{u}} , \quad e^{\dot{+}}_{\dot{+}} = k\partial_{\bar{u}} , \quad e^{\dot{+}}_{\dot{-}} = k^{-1}\partial_z - kK_{z\bar{u}}\partial_u , \quad e^{\dot{-}}_{\dot{-}} = k\partial_u \quad (7.13)$$

We now construct a rank $2n$ Killing spinor for an $\mathcal{O}(2n)$ multiplet η . The components of η are related to the components of the spinor by:

$$\eta_i = \binom{2n}{i} \underbrace{\eta_{+\dots+}}_{2n-i} \underbrace{\eta_{-\dots-}}_i , \quad \eta \equiv \sum_0^{2n} \eta_i \zeta^i \quad (7.14)$$

The Killing spinor equation is

$$e^{\dot{A}}_{(A} \eta_{B_1 \dots B_{2n})} = 0 \quad (7.15)$$

because we work in a frame where the connection 1-form ω^A_B vanishes, or

$$e^{\dot{A}}_{-} \eta_{i-1} + e^{\dot{A}}_{+} \eta_i = 0 . \quad (7.16)$$

We begin by checking $i = 0, 1$. In the generalized Legendre transform construction above, we identify⁹

$$\eta_0 = z , \quad \eta_1 = -K_u , \quad \eta_{2n} = (-1)^n \bar{z} , \quad \eta_{2n-1} = (-1)^n K_{\bar{u}} . \quad (7.17)$$

Then (7.16) is trivially satisfied for $i = 0$. For $i = 1$, we have:

$$e^{\dot{+}}_{-} z - e^{\dot{+}}_{+} K_u = k^{-1} - kK_{u\bar{u}} = 0 , \quad e^{\dot{-}}_{-} z - e^{\dot{-}}_{+} K_u = 0 - k^{-1}K_{u\bar{z}} + kK_{u\bar{z}}K_{u\bar{u}} = 0 . \quad (7.18)$$

⁹In [6] and many other references, the role of z, u is interchanged with \bar{z}, \bar{u} ; also, in some references, the η 's are defined with an extra overall factor ζ^{-n} .

The $i = 2n, 2n+1$ equations are just the complex conjugates of the above. For $1 < i < 2n$, we find equations that do not have a simple expression in terms of the Kähler-potential; however, we can easily prove that they are satisfied by studying the superspace description of the $\mathcal{O}(2n)$ multiplet η . The superspace constraints (2.5) can be written as

$$D_1^\alpha \eta_{i-1} + D_2^\alpha \eta_i = 0 , \quad (7.19)$$

where D_a^α are the superspace spinor derivatives with isospin indices a and spinor indices α . Note the similarity to (7.16). For $i = 0, 1, 2n, 2n+1$, (7.19) is a set of relations between $D_a^\alpha x^\mu$, where $x^\mu = \{z, u, \bar{z}, \bar{u}\}$. Note that these relations are exactly the same as those obeyed by $e_\pm^\alpha x^\mu$ as a consequence of (7.16). In superspace, however, (7.19) is imposed as a constraint that defines η . When we eliminate the $N = 1$ auxiliary superfields $\eta_i, 1 < i < 2n-1$, and the Legendre transform variables η_1, η_{2n-1} , we must consider $\eta_i(x^\mu)$. Then the equations (7.19) become:

$$\partial_\mu \eta_{i-1} D_1^\alpha x^\mu + \partial_\mu \eta_i D_2^\alpha x^\mu = 0 . \quad (7.20)$$

However, since the linear relations between the $D_a^\alpha x^\mu$ and the $e_\pm^\alpha x^\mu$ are the same, this implies relations between the $\partial_\mu \eta_{i-1}$ and $\partial_\mu \eta_i$ that guarantee that the Killing spinor equation (7.16) is satisfied.

The leading component of the Killing spinors discussed here is proportional to a coordinate; there is a closely related Killing tensor that can be constructed out of the spinors which may be easier to define globally. This is defined by the components of the derivative of the Killing spinor that do not vanish:

$$X_{B_1 \dots B_{2n-1}}^{\dot{A}} \equiv \nabla^{A\dot{A}} \eta_{AB_1 \dots B_{2n-1}} . \quad (7.21)$$

Because the connection is self-dual, these obey the Killing tensor equations [25]

$$\nabla_{\dot{A}}^{B_1} X_{B_1 \dots B_{2n-1}}^{\dot{A}} = 0 , \quad \nabla_{(B_1}^{(\dot{B}} X_{B_2 \dots B_{2n})}^{\dot{A})} = 0 . \quad (7.22)$$

For $n = 1$, this is the well-known triholomorphic Killing vector that characterizes the $\mathcal{O}(2)$ geometries [26].

7.3 Higher dimensional hyperkähler manifolds

For four dimensional hyperkähler manifolds, we were able to explicitly relate projective superspace and geometry; bolstered by our success, we can conjecture geometric results from projective superspace for the higher dimensional case: In projective superspace, higher dimensional target spaces arise when one considers models with more independent superfields. Depending on the type of multiplets in the model, we will get corresponding Killing spinors and Killing tensors.

8 Properties of twistor space

For the reader's convenience, we review the properties of twistor spaces summarized in section 1.1 and relate them to the geometric structure that projective superspace revealed.

The description of hyperkähler geometry that follows from the projective superspace formulation of $N = 2$ supersymmetric σ -models leads to a coherent picture in twistor space, where the \mathbb{P}^1 of graded abelian subalgebra of the $N = 2$ superalgebra is identified with the \mathbb{P}^1 of complex structures on the hyperkähler manifold. The fundamental object is the 2-form Ω (5.1). In terms of the hyperkähler structure, it can be written as:

$$\Omega = \omega^{(2,0)} + \zeta \omega^{(1,1)} - \zeta^2 \omega^{(0,2)} , \quad (8.1)$$

where $\omega^{(2,0)}$ is a nondegenerate holomorphic 2-form and $\omega^{(1,1)}$ is the Kähler form with respect to the complex structure at the north pole of the \mathbb{P}^1 . One may always choose Darboux coordinates z, u for the holomorphic symplectic structure $\omega^{(2,0)}$; extending these to arbitrary complex structures parametrized by a point ζ on the \mathbb{P}^1 lifts z, u to $\Upsilon(\zeta), \tilde{\Upsilon}(\zeta)$ and leads us to write

$$\Omega(\zeta) = id\Upsilon d\tilde{\Upsilon} , \quad (8.2)$$

with $\Upsilon(\zeta), \tilde{\Upsilon}(\zeta)$ such that $\Omega(\zeta)$ is projectively real, and hence a section of $\mathcal{O}(2) \otimes \Omega^2(M)$. The reality condition implies the existence of a twisted symplectomorphism from the north pole to the south pole, and consequently the existence of the generating function $f(\Upsilon, \tilde{\Upsilon}; \zeta)$. This in particular *proves* that the projective superspace formalism with polar superfields $\Upsilon, \tilde{\Upsilon}$ is completely general (at least locally in each patch of the hyperkähler manifold, though we see no obstruction to patching this together over the whole manifold using the general symplectic transformations of section 6).

An interesting feature of this way of thinking about hyperkähler geometry is that it naturally leads to two separate problems: (1) What is $f(\Upsilon, \tilde{\Upsilon}; \zeta)$? and (2) What is $\Upsilon(\zeta)$? In $N = 2$ language, the first is an off-shell problem and the second is the on-shell problem. It may be possible to solve the off-shell problem for, *e.g.*, $K3$, without solving the on-shell problem. This would still be very interesting, though it would not yield an explicit metric.

The 2-form Ω also allows us to find the system of partial differential equations that characterize hyperkähler geometry. For a $4D$ -dimensional hyperkähler manifold M , the form (8.2) clearly obeys

$$\Omega^{D+1} = 0 . \quad (8.3)$$

For $D = 1$, this reduces to the usual Monge-Ampère equation. For higher D , this gives a nice system of equations that implies the Monge-Ampère equation. For example, for $D = 2$, expanding in ζ , we find

$$\omega^{(2,0)}((\omega^{(1,1)})^2 - \omega^{(2,0)}\omega^{(0,2)}) = 0 ,$$

$$\begin{aligned}\omega^{(1,1)}((\omega^{(1,1)})^2 - 6\omega^{(2,0)}\omega^{(0,2)}) &= 0 , \\ \omega^{(0,2)}((\omega^{(1,1)})^2 - \omega^{(2,0)}\omega^{(0,2)}) &= 0 .\end{aligned}\tag{8.4}$$

This implies the Monge-Ampère equation, which in our conventions for general dimension D is

$$(\omega^{(1,1)})^{2D} - \binom{2D}{D} (\omega^{(2,0)}\omega^{(0,2)})^D = 0 .\tag{8.5}$$

9 Rotating the complex structures

A crucial role both for the twistor structure and for the supersymmetric σ -models is played by rotations of the \mathbb{P}^1 combined with corresponding rotations of the hyperkähler structure on M . We consider the 2-form Ω with $\omega^{(2,0)}$ in Darboux coordinates $\omega^{(2,0)} = \frac{i}{2}\epsilon_{ij}dz^i dz^j$:

$$\Omega(\zeta) = idzdu + i\partial\bar{\partial}K\zeta + id\bar{z}d\bar{u}\zeta^2 ,\tag{9.1}$$

where $\partial\bar{\partial}K = K_{z\bar{z}}dzd\bar{z} + K_{z\bar{u}}dzd\bar{u} + K_{u\bar{z}}dud\bar{z} + K_{u\bar{u}}dud\bar{u}$. As described in previous sections of this article, the form Ω is a real section of an $O(2)$ bundle, where the real structure is defined by complex conjugation composed with the antipodal map $\bar{\zeta} \rightarrow -1/\zeta$, and acts on an $O(2n)$ section $\eta = \sum_0^{2n} \zeta^i \eta_i$ as:

$$\eta(\zeta) = (-)^n \zeta^{2n} \bar{\eta}(\frac{-1}{\zeta}) .\tag{9.2}$$

For the $\mathcal{O}(2)$ case, we have

$$\eta_0 = -\bar{\eta}_2, \quad \eta_1 = \bar{\eta}_1 .\tag{9.3}$$

9.1 Rotating \mathbb{P}^1

An $SU(2)$ R-symmetry transformation in superspace is generated by Möbius transformations of ζ , and rotates the complex structures on the hyperkähler manifold. If we write

$$\zeta' = \frac{a\zeta + b}{c\zeta + d} ,\tag{9.4}$$

where $ad - bc = 1$ and $\bar{d} = a, \bar{c} = -b$ for an $SU(2)$ transformation, then for $a = 1 + i\alpha, b = \beta$, and $\alpha = \bar{\alpha}$, the infinitesimal transformation of ζ is

$$\delta\zeta = \beta + 2i\alpha\zeta + \bar{\beta}\zeta^2 .\tag{9.5}$$

An $SU(2)$ -transformation is generated by

$$\underline{\alpha} \cdot \underline{J} \equiv \sum_1^3 \alpha_i J_i \equiv \alpha J_3 + \frac{1}{2}\beta J_- + \frac{1}{2}\bar{\beta} J_+ ,\tag{9.6}$$

where the $SU(2)$ -algebra is

$$J_{\pm} \equiv J_1 \pm iJ_2, \quad [J_3, J_{\pm}] = \pm J_{\pm}, \quad [J_+, J_-] = 2J_3. \quad (9.7)$$

Writing

$$\delta\zeta = [2i\underline{\alpha} \cdot \underline{J}, \zeta] \quad (9.8)$$

we may represent the $SU(2)$ generators as

$$J_- = -i\partial_{\zeta}, \quad J_3 = \zeta\partial_{\zeta}, \quad J_+ = -i\zeta^2\partial_{\zeta}. \quad (9.9)$$

More generally, we can add a spin piece, and write

$$J_- = -i\partial_{\zeta}, \quad J_3 = \zeta\partial_{\zeta} - \frac{1}{2}h, \quad J_+ = -i\zeta^2\partial_{\zeta} + ih\zeta. \quad (9.10)$$

An $\mathcal{O}(2n)$ multiplet transforms with $h = 2n$, and Ω transforms with $h = 2$ (see, *e.g.*, [3] and [11]). Then, from $\delta\Omega = -2i(\alpha J_3 + \frac{1}{2}\beta J_- + \frac{1}{2}\bar{\beta} J_+)\Omega$, we find

$$\delta(idzdu) = 2i\alpha(idzdu) - \beta(i\partial\bar{\partial}K), \quad (9.11)$$

$$\delta(i\partial\bar{\partial}K) = -2\beta(id\bar{z}d\bar{u}) + 2\bar{\beta}(idzdu). \quad (9.12)$$

$$\delta(id\bar{z}d\bar{u}) = -2i\alpha(id\bar{z}d\bar{u}) + \bar{\beta}(i\partial\bar{\partial}K), \quad (9.13)$$

9.2 Rotating the hyperkähler structure on M

It is easy to find diffeomorphisms on M that satisfy (9.11):

$$\delta z = i\alpha h z + \beta K_u, \quad \delta u = i\alpha(2 - h)u - \beta K_z \quad (9.14)$$

clearly give the correct transformation. Notice the close relation to the Legendre transform construction: $-K_u \equiv \eta_1$, so $\delta z = i\alpha h z - \beta\eta_1$. This is exactly what we would expect from projective superspace; by changing h , we get different η and or Υ multiplets. As the α transformations are holomorphic, ∂ and $\bar{\partial}$ are invariant under them. Naively, K transforms as $\delta_{\alpha}K = i\alpha[hzK_z + (2 - h)uK_u]$; we can cancel this by simply subtracting this from the variation of K ; thus we define $\delta_{\alpha}K = i\alpha[hzK_z + (2 - h)uK_u] + \delta'_{\alpha}K = 0$. This may look odd, but as we shall see, it is very necessary and much more nontrivial below.

Thus we focus on the β transformations. We write them as

$$\delta_{\beta}z^i = \beta\epsilon^{ij}K_j, \quad \delta_{\beta}\bar{z}^i = 0, \quad (9.15)$$

where $\{z^i\} \equiv \{u, z\}$. Note that here the naive variation of K vanishes: $\delta_{\beta}K = \beta\epsilon^{ij}K_jK_i = 0$. Consequently, we have:

$$\delta_{\beta}(i\partial\bar{\partial}K) = i[d(\delta_{\beta}z^i)\partial_i\bar{\partial}K + dz^i(\delta_{\beta}\partial_i)\bar{\partial}K + \partial dz^i(\delta_{\beta}\bar{\partial}_i)K + \partial\bar{\partial}(\delta'_{\beta}K)]. \quad (9.16)$$

Because $\delta_\beta \bar{z}^i = 0$, we have $\delta_\beta \partial_i = -(\partial_i \delta_\beta z^j) \partial_j$, *etc.*, and we find

$$\begin{aligned}
\delta_\beta(i\partial\bar{\partial}K) &= i[d(\delta_\beta z^i)\partial_i\bar{\partial}K - dz^i(\partial_i\delta_\beta z^j)\partial_j\bar{\partial}K - \partial(d\bar{z}^i(\bar{\partial}_i\delta_\beta z^j)K_j) + \partial\bar{\partial}(\delta'_\beta K)] \\
&= i[(\partial\delta_\beta z^i)\partial_i\bar{\partial}K + (\bar{\partial}\delta_\beta z^i)\partial_i\bar{\partial}K - (\partial\delta_\beta z^i)\partial_i\bar{\partial}K - \partial((\bar{\partial}\delta_\beta z^i)K_i) + \partial\bar{\partial}(\delta'_\beta K)] \\
&= i[(\bar{\partial}\delta_\beta z^i)\partial_i\bar{\partial}K - \partial((\bar{\partial}\delta_\beta z^i)K_i) + \partial\bar{\partial}(\delta'_\beta K)] \\
&= i[(\bar{\partial}\delta_\beta z^i)\bar{\partial}K_i - \partial((\bar{\partial}\delta_\beta z^i)K_i) + \partial\bar{\partial}(\delta'_\beta K)] .
\end{aligned} \tag{9.17}$$

Now we substitute (9.15):

$$i(\bar{\partial}\delta_\beta z^i)\bar{\partial}K_i = i\beta\epsilon^{ij}(\bar{\partial}K_j)\bar{\partial}K_i = i\beta\epsilon^{ij}d\bar{z}^j d\bar{z}^i = -2i\beta d\bar{u}d\bar{z} , \tag{9.18}$$

where we use the quaternionic relation $\omega^{(1,1)}[\omega^{(2,0)}]^{-1}\omega^{(1,1)} = -\omega^{(0,2)}$. Finally, we need to show that all remaining terms can cancel. In contrast to (9.18), which is a $(2, 0)$ form, the remaining terms $i[-\partial((\bar{\partial}\delta_\beta z^i)K_i) + \partial\bar{\partial}(\delta'_\beta K)]$ are both $(1, 1)$ forms. We need to show that $\partial((\bar{\partial}\delta_\beta z^i)K_i)$ is both ∂ and $\bar{\partial}$ closed; this is manifest for ∂ ; For $\bar{\partial}$, we use (9.18):

$$\bar{\partial}\partial((\bar{\partial}\delta_\beta z^i)K_i) = \partial((\bar{\partial}\delta_\beta z^i)\bar{\partial}K_i) = \partial(-2\beta d\bar{u}d\bar{z}) = 0 . \tag{9.19}$$

Thus there exists a $\delta'_\beta K$ such that the total variation $\delta_\beta(i\partial\bar{\partial}K)$ is given by (9.12).

9.3 The Kähler potential is a Hamiltonian

A remarkable feature allows us to interpret the Kähler potential K as a Hamiltonian function. The transformation (9.5) has a fixed point at $\zeta = \pm i$ for $\alpha = 0, \beta = \bar{\beta}$; then (9.11,9.12,9.13) imply that $\delta_0 \equiv \delta_{\alpha=0, \beta=\bar{\beta}}$ preserves

$$[\omega^{(2,0)} + \omega^{(0,2)}] = \frac{1}{2}[\Omega(\zeta = i) + \Omega(\zeta = -i)] ; \tag{9.20}$$

Thus δ_0 is a symplectomorphism that preserves $\text{Re}(\omega^{(2,0)})$, and hence is generated by a moment map; this moment map is precisely the i times the Kähler potential:

$$[\omega^{(2,0)} + \omega^{(0,2)}](\delta_0 z^i, .) = idK . \tag{9.21}$$

This generalizes the observation in [14] that for manifolds with an isometry that rotates the complex structure, the Kähler potential can be viewed as the moment map of the rotation with respect to a complex structure preserved by the rotation; *here we do not need an isometry.*

10 Normal gauge

On any Kähler manifold, one can define a *normal gauge* for the Kähler potential [27]. In this gauge, one eliminates any purely holomorphic or antiholomorphic pieces using Kähler

transformations, and uses holomorphic coordinate transformations to make the potential as close as possible to flat:

$$K = z^i \bar{z}^i + \mathcal{O}(z^2 \bar{z}^2) , \quad (10.1)$$

i.e., all terms except for the flat term are at least quadratic in z and quadratic in \bar{z} ; these terms are all expressible in terms of the curvature and its derivatives, and the explicit expression is easily found by direct computation. Clearly, normal gauge is unique up to the choice of base point, and up to constant $U(2)$ transformations.

For a Ricci-flat Kähler manifold,

$$\det g_{i\bar{j}} = f(z) \bar{f}(\bar{z}) ; \quad (10.2)$$

in normal gauge, $f(z)$ is constant, as follows from (10.1), which implies

$$(\partial_z)^n g_{i\bar{j}}|_{(z=\bar{z}=0)} = (\partial_{\bar{z}})^n g_{i\bar{j}}|_{(z=\bar{z}=0)} = 0 \quad \forall n . \quad (10.3)$$

For a hyperkähler manifold (at least for real $D=4$), we have $(\omega^1)^2 = (\omega^2)^2 = (\omega^3)^2 \propto \det g_{i\bar{j}}$, and hence $\omega^{(2,0)} \omega^{(0,2)}$ is constant. However, since $\omega^{(2,0)}$ is holomorphic, and its magnitude is constant, we conclude that it is in Darboux coordinates (up to a constant phase which can be absorbed by a constant $U(1)$ transformation that preserves the normal gauge); thus:

$$\omega^{(2,0)} = i dz^1 dz^2 . \quad (10.4)$$

11 Example: the Eguchi-Hansen geometry

In this section we derive the Eguchi-Hansen metric using the methods developed above. This related to the general program of constructing hyperkähler metrics on cotangent bundles of symmetric spaces using projective superspace methods [28, 29, 30, 31], and indeed can be applied to all of them. Other recent examples in the projective/twistor formalism include the explicit elliptic examples of [32] and the explicit linear deformations of hyperkähler manifolds given in [33].

The Eguchi-Hansen metric lives on the cotangent space \mathbb{P}^1 ; hence we start with the Fubini-Study Kähler potential for \mathbb{P}^1 and lift it to $\mathcal{N} = 2$ superspace:

$$f = \ln(1 + \Upsilon \bar{\Upsilon}) . \quad (11.1)$$

The Eguchi-Hansen metric has a triholomorphic $SU(2)$ isometry which can be realized by $PSU(2)$ transformations of Υ . We can therefore choose a particular form for Υ and reach general points by acting with the isometry [29]. In particular, we make the ansatz that when we set $z \equiv \Upsilon(0) = 0$ then

$$\Upsilon|_{z=0} = y\zeta \quad (11.2)$$

is a valid point on the manifold. We now act by a $PSU(2)$ transformation which we parameterize so as to recover (11.2) as well as $z = \Upsilon(0)$:

$$\Upsilon \rightarrow \frac{z + y\zeta}{1 - y\bar{z}\zeta} ; \quad (11.3)$$

note that this is a triholomorphic $PSU(2)$ transformation that acts on Υ , *not* a rotation of the \mathbb{P}^1 of complex structures parameterized by ζ . The conjugate is

$$\bar{\Upsilon} = \frac{\bar{y} - \bar{z}\zeta}{z\bar{y} - \zeta} . \quad (11.4)$$

Following the methods described above, to find Ω we need to calculate $\tilde{\Upsilon}$:

$$\tilde{\Upsilon} = \zeta \frac{\partial f}{\partial \Upsilon} = \frac{\zeta \tilde{\Upsilon}}{1 + \Upsilon \tilde{\Upsilon}} = \frac{(\bar{y} - \bar{z}\zeta)(1 - y\bar{z}\zeta)}{(1 + z\bar{z})(1 - y\bar{y})} . \quad (11.5)$$

A quick calculation reveals that $i d\Upsilon d\tilde{\Upsilon}$ is indeed a section of $\mathcal{O}(2)$; the structure is clarified if we introduce the second holomorphic coordinate

$$u \equiv \tilde{\Upsilon}(0) = \frac{\bar{y}}{(1 + z\bar{z})(1 - y\bar{y})} , \quad (11.6)$$

which implies

$$y = \frac{2(1 + z\bar{z})\bar{u}}{1 + \sqrt{1 + 4u\bar{u}(1 + z\bar{z})^2}} . \quad (11.7)$$

This gives the standard Ω for the Eguchi-Hansen Kähler form:

$$\Omega_{EH} = i d\Upsilon d\tilde{\Upsilon} = i dz du + \zeta \omega_{EH}^{(1,1)} + i \zeta^2 d\bar{z} d\bar{u} \quad (11.8)$$

where

$$\omega_{EH}^{(1,1)} = -i \frac{1 + z\bar{z}}{\sqrt{1 + 4u\bar{u}(1 + z\bar{z})^2}} \left(dud\bar{u} + \frac{dzd\bar{z}}{(1 + z\bar{z})^3} + (zdu + 2udz)(\bar{z}d\bar{u} + 2\bar{u}d\bar{z}) \right) . \quad (11.9)$$

This can be made more familiar by the holomorphic symplectomorphism

$$u = \frac{1}{2}u'^2 , \quad z = \frac{z'}{u'} \quad (11.10)$$

which gives

$$\begin{aligned} \omega_{EH}^{(1,1)} &= -i \frac{1}{\sqrt{1 + r^4}} \left(r^2 (du' d\bar{u}' + dz' d\bar{z}') + \frac{1}{r^4} (z' du' - u' dz') (\bar{z}' d\bar{u}' - \bar{u}' d\bar{z}') \right) , \\ r &\equiv \sqrt{u' \bar{u}' + z' \bar{z}'} . \end{aligned} \quad (11.11)$$

This calculation reveals an important feature of our approach and the virtue of using Ω : we found the Kähler-form without evaluating any contour integral; in particular, there are no ambiguities about the orientation of the contour that can arise in a direct evaluation of the superspace Lagrangian. An example of such issues is given in Appendix B.

12 Outlook

We have discussed the intimate relation between twistor space and supersymmetry as manifested in projective superspace.

Our primary tools are $N = 2$ sigma models with hyperkähler target spaces, but gauging them also introduces gauge connections. These were mainly used here to describe quotient constructions and dualities, but may be studied in their own right in projective superspace. This leaves one obvious gap in the description of models: $N = 2$ supergravity. To a certain extent this gap is presently being filled (see [34] and references therein.)

A more immediate extension of the framework presented here is to include quaternionic Kähler manifolds. Such an extension is presently under way.

We further note that projective superspace has recently been used to study linear perturbations of a class of hyperkähler metric in [33], where an extension to quaternionic Kähler metrics is also advertised. As our description is fully non-linear, a comparison should be fruitful.

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A The hyperkähler quotient in projective superspace

For completeness we review constructions having to do with gauge fields, quotients, and dualities in projective superspace. The hyperkähler quotient construction was discovered in [35] and its geometric interpretation was given in [14]. The tools to describe it in projective superspace were developed in [8], and the description was given in [20], though it has been known to us for a long time. Here we review it.

A.1 Isometries

The polar multiplet Υ has an infinite number of $N = 1$ superfields; consequently, it is difficult to extract the Kähler potential except in special circumstances. On the other hand, the space of polar multiplets has an algebraic structure: holomorphic functions of arctic multiplets are themselves arctic. This allows for a very direct realization of triholomorphic isometries of the hyperkähler geometry in projective superspace: they are simply symmetries of the projective superspace action (2.9) that are holomorphic in the arctic multiplets.

As we explain below, the whole process of gauging triholomorphic isometries and performing hyperkähler quotients, when described in terms of polar multiplets in projective superspace is essentially the same procedure as for Kähler quotients described in terms of chiral superfields in $N = 1$ superspace [36],[14].

A triholomorphic isometry acts without rotating the complex structures; therefore it is generated by a holomorphic vector field $X(\Upsilon)$ that has no explicit dependence on ζ :

$$\delta\Upsilon = X(\Upsilon) , \quad \delta\bar{\Upsilon} = \bar{X}(\bar{\Upsilon}) . \quad (\text{A.1})$$

When we gauge a symmetry generated by such a vector field, we introduce a local parameter $\lambda(\zeta)$:

$$\delta\Upsilon = \lambda(\zeta)X(\Upsilon) , \quad \delta\bar{\Upsilon} = \bar{\lambda}(\frac{-1}{\zeta})\bar{X}(\bar{\Upsilon}) ; \quad (\text{A.2})$$

to preserve the holomorphic properties of Υ , the parameter $\lambda(\zeta)$ must itself be an arctic superfield, and consequently, $\bar{\lambda}(\frac{-1}{\zeta})$ must be antarctic. We are thus led to introduce a real tropical field $\mathbf{V} = \Re(\mathbf{V})$; it has coefficients for all powers of ζ that are unconstrained as $N = 1$ superfields. It transforms as

$$\delta\mathbf{V} = i(\bar{\lambda} - \lambda) . \quad (\text{A.3})$$

This may be generalized to a nonabelian action, where $\mathbf{V}, \lambda, \bar{\lambda}$ all become matrix valued; for a finite transformation by an element $g = e^{\mathcal{L}_{i\lambda X}}$, we have:

$$(e^{\mathbf{V}}) = e^{i\bar{\lambda}} e^{\mathbf{V}} e^{-i\lambda} . \quad (\text{A.4})$$

Having introduced the field \mathbf{V} , we now show how it describes $N = 2$ super Yang-Mills theory [8]. We split the tropical gauge multiplet factors regular at the north and south poles:

$$e^{\mathbf{V}} = e^{\mathbf{V}^-} e^{\mathbf{V}^+} , \quad \mathbf{V}_+ = \sum_{n=0}^{\infty} \mathbf{V}_{+n} \zeta^n , \quad \mathbf{V}_- = \bar{\mathbf{V}}_+ . \quad (\text{A.5})$$

Because \mathbf{V} is an analytic superfield, $\nabla e^{\mathbf{V}} = 0$, and we may define a gauge-covariant analytic derivative \mathcal{D}

$$\mathcal{D} \equiv \nabla + e^{-\mathbf{V}^-} (\nabla e^{\mathbf{V}^-}) = \nabla - (\nabla e^{\mathbf{V}^+}) e^{-\mathbf{V}^+} . \quad (\text{A.6})$$

Comparing powers of ζ for both expressions, we conclude that \mathbf{D} has only a constant and a linear term (just as ∇), and hence defines the $N = 2$ gauge-covariant derivative (for a more detailed explanation see [8]). This structure is precisely the same as Ward's twistor construction of self-dual Yang-Mills fields [37]. Observe that (A.6) depends crucially on the reality of \mathbf{V} ;

We find the covariantly chiral gauge field strength \mathbf{W} by computing

$$\{\bar{\mathcal{D}}_{\dot{\alpha}}(\zeta), \frac{\partial}{\partial \zeta}(\bar{\mathcal{D}}_{\dot{\beta}}(\zeta))\} = \varepsilon_{\dot{\alpha}\dot{\beta}} \mathbf{W} . \quad (\text{A.7})$$

Note that \mathbf{W} is ζ independent.

We focus on the case when we start with a vector space, and quotient by a linear (or possibly affine) action; this has the virtue that the formal expression (2.8) for the superspace Lagrangian can be explicitly evaluated. Thus we start with

$$f(\Upsilon, \bar{\Upsilon}, \mathbf{V}) = \bar{\Upsilon} e^{\mathbf{V}} \Upsilon \quad (\text{A.8})$$

for any compact group acting linearly on the vector space coordinatized by Υ .

We define *covariantly* analytic polar multiplets

$$\hat{\Upsilon} \equiv e^{\mathbf{V}+} \Upsilon , \quad \hat{\bar{\Upsilon}} = \bar{\Upsilon} e^{\mathbf{V}-} . \quad (\text{A.9})$$

In terms of these, the gauge-invariant Lagrange density (A.8) is quadratic; hence, the ζ integral

$$F = \oint_C \frac{d\zeta}{2\pi i \zeta} \bar{\Upsilon} e^{\mathbf{V}} \Upsilon = \oint_C \frac{d\zeta}{2\pi i \zeta} \hat{\bar{\Upsilon}} \cdot \hat{\Upsilon} \quad (\text{A.10})$$

can be trivially evaluated, and the auxiliary superfields can be integrated out to get the gauge-invariant $N = 1$ superspace Lagrangian

$$L_{N=1} = \hat{\hat{z}} \cdot \hat{z} - \hat{\hat{s}} \cdot \hat{s} , \quad (\text{A.11})$$

where $\hat{z} \equiv \hat{\Upsilon}_0$ are $N = 1$ gauge-covariantly (vector representation) chiral superfields and $\hat{s} \equiv \hat{\Upsilon}_1$ are modified $N = 1$ gauge-covariantly complex linear superfields

$$\bar{\mathcal{D}}_{\dot{\alpha}} \hat{z} = 0 , \quad \bar{\mathcal{D}}^2 \hat{s} = \hat{W} \hat{z} . \quad (\text{A.12})$$

Here \hat{W} is the $N = 1$ covariantly chiral projection of the $N = 2$ field strength \mathbf{W} (A.7) in the representation that acts on \hat{z} and \mathcal{D} is the $N = 1$ gauge-covariant derivative. We can go to chiral representation and replace $\hat{z}, \hat{s}, \hat{W}$ with ordinary chiral and linear superfields z, s, W by introducing the $N = 1$ gauge potential V :

$$e^V \equiv e^{V-} e^{V+} , \quad \hat{z} = e^{V+} z , \quad \hat{s} = e^{V+} s , \quad \hat{W} = e^{V+} W e^{-V+} , \quad (\text{A.13})$$

where $V_{\pm} \equiv \mathbf{V}_{0\pm}$ is the $N = 1$ projection of the ζ -independent coefficients of \mathbf{V}_{\pm} . These substitutions lead to

$$L_{N=1} = \bar{z} e^V z - \bar{s} e^V s , \quad (\text{A.14})$$

$$\bar{D}^2 s = W z . \quad (\text{A.15})$$

It is convenient to rewrite the $N = 1$ Lagrangian (A.14) in terms of chiral superfields; to do this, we impose the constraints (A.15) by chiral Lagrange multipliers u in a superpotential term

$$u(\bar{D}^2 s - W z) , \quad (\text{A.16})$$

and integrate out s to obtain the nonabelian generalization of the $N = 1$ gauged Lagrangian (after relabeling $z \rightarrow z_+, u \rightarrow z_-$):

$$L_{N=1} = \bar{z}_+ e^V z_+ - z_- e^{-V} \bar{z}_- . \quad (\text{A.17})$$

In addition, we are left with a superpotential term

$$\text{Tr} [W \mu^+] = z_- W z_+ , \quad (\text{A.18})$$

where μ^+ is just the holomorphic moment map. Observe that interchanging $z_+ \leftrightarrow z_-$ and changing the representation of V to its conjugate does not modify the gauged Lagrangian (A.17); this implies that in the original $N = 2$ Lagrangian F (A.10), we can take Υ transforming in the conjugate representation (*e.g.*, opposite charge for $U(1)$) without changing the final result. This interchange can be implemented directly in projective superspace by the $\Upsilon \leftrightarrow \tilde{\Upsilon}$ duality transformation of Section 6 ($\tilde{\Upsilon}$ naturally transforms in the conjugate representation to Υ). In the next subsection we integrate out the $N = 2$ gauge fields to find the quotient Lagrangian; in $N = 1$ superspace, integrating out the chiral superfield W imposes the moment map constraint $\mu^+ = 0$.

A.2 Quotients and Duality

Just as $N = 2$ isometries and gauging in projective superspace bear a striking resemblance to their $N = 1$ superspace analogs, so do $N = 2$ quotients and duality; indeed, the tensor multiplet projective superspace Lagrangian is just the Legendre transform of the polar multiplet Lagrangian.

The procedure we follow is the same as in $N = 1$ superspace: we gauge the relevant isometries as above; to perform a quotient, we simply integrate out the gauge prepotential e^V . Since this does not break the isometry, we are left with an action defined on the quotient space. To find the dual, we add a Lagrange multiplier η that constrains the gauge

prepotential to be trivial¹⁰, and again integrate out \mathbf{V} ; the dual field is then the Lagrange multiplier η . As in the $N = 1$ case, we only consider duality for abelian isometries. In that case, the Lagrange multiplier term that constrains \mathbf{V} is

$$\frac{\eta}{\zeta} \mathbf{V} , \quad (\text{A.19})$$

where η is the $O(2)$ superfield that describes the $N = 2$ tensor multiplet [3].

B Dualities and contour ambiguities

The Eguchi-Hansen metric can also be described in terms of the $\mathcal{O}(2)$ -multiplet [3] (these are particular instances of the multiplets described in Section 7). A particularly nice way of finding this description involves the quotient and duality described in the previous appendix. Starting from (11.1), one can write

$$f_{\mathbf{V}} = \ln(1 + e^{\mathbf{V}}) - \frac{\eta_{(2)}}{\zeta} \mathbf{V} \quad (\text{B.20})$$

where $\eta_{(2)}$ is an $\mathcal{O}(2)$ -multiplet; eliminating $\eta_{(2)}$ imposes the condition that $\mathbf{V} \propto \ln(\Upsilon \bar{\Upsilon})$, whereas eliminating \mathbf{V} gives:

$$f_{\eta} = -\frac{\eta_{(2)}}{\zeta} \ln \frac{\eta_{(2)}}{\zeta} - \left(1 - \frac{\eta_{(2)}}{\zeta}\right) \ln \left(1 - \frac{\eta_{(2)}}{\zeta}\right) . \quad (\text{B.21})$$

The metric can be found by evaluating the ζ integral along a contour first given in [3] with the caveat that the opposite orientation must be used for the two terms in (B.21) to obtain a metric with definite signature.

On the other hand, we can rewrite (11.1) in terms of the symplectic conjugate variables $\tilde{\Upsilon}$:

$$\tilde{f} = \ln \left(1 + \sqrt{1 - 4\tilde{\Upsilon} \bar{\tilde{\Upsilon}}}\right) - \sqrt{1 - 4\tilde{\Upsilon} \bar{\tilde{\Upsilon}}} . \quad (\text{B.22})$$

Performing the duality transformation to the $\mathcal{O}(2)$ multiplet $\eta_{(2)}$ as above, we obtain:

$$\tilde{f}_{\eta} = -\frac{\eta_{(2)}}{\zeta} \ln \frac{\eta_{(2)}}{\zeta} - \left(1 + \frac{\eta_{(2)}}{\zeta}\right) \ln \left(1 + \frac{\eta_{(2)}}{\zeta}\right) . \quad (\text{B.23})$$

The difference in relative sign between the terms in (B.21) and (B.23) mean that we need to use different orientations of the contours when evaluating the metric in the two cases. Clearly the issue of contours, in particular their orientation, is a subtle one. In the definition of Ω no ambiguities exist, as illustrated in Sec. 11. We thus determine the integration contours by requiring agreement with an Ω derivation. It would be interesting to compare this idea to the discussions of contours presented in [23, 33].

¹⁰As explained in [14, 38], this is the correct geometric way of understanding duality; when one chooses coordinates such that the killing vectors generating the isometries are constant, this gives the usual Legendre transform.

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